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ENGINEERING COLLEGE
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Engineering Mechanics
Statics Lectures

Chapter five

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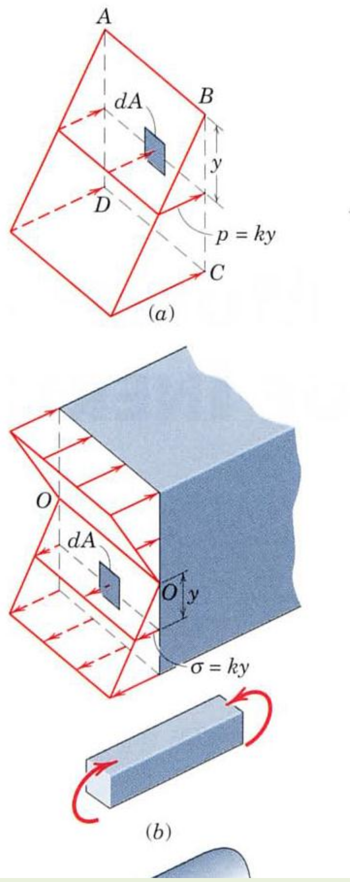
Chapter five

5-1 Area Moment of inertia.

A/1 INTRODUCTION

When forces are distributed continuously over an area on which they act, it is often necessary to calculate the moment of these forces about some axis either in or perpendicular to the plane of the area. Frequently the intensity of the force (pressure or stress) is proportional to the distance of the line of action of the force from the moment axis. The elemental force acting on an element of area, then, is proportional to distance times differential area, and the elemental moment is proportional to distance squared times differential area. We see, therefore, that the total moment involves an integral of form $\int (\text{distance})^2 \, d(\text{area})$. This integral is called the *moment of inertia* or the *second moment* of the area. The integral is a function of the geometry of the area and occurs frequently in the applications of mechanics. Thus it is useful to develop its properties in some detail and to have these properties available for ready use when the integral arises.

Figure A/1 illustrates the physical origin of these integrals. In part *a* of the figure, the surface area *ABCD* is subjected to a distributed pressure p whose intensity is proportional to the distance y from the axis *AB*. This situation was treated in Art. 5/9 of Chapter 5, where we described the action of liquid pressure on a plane surface. The moment about *AB* due to the pressure on the element of area dA is $py\,dA = ky^2\,dA$. Thus, the integral in question appears when the total moment $M = k \int y^2\,dA$ is evaluated.



In Fig. A/1*b* we show the distribution of stress acting on a transverse section of a simple elastic beam bent by equal and opposite couples applied to its ends. At any section of the beam, a linear distribution of force intensity or stress σ, given by $\sigma = ky$, is present. The stress is positive (tensile) below the axis *O–O* and negative (compressive) above the axis. We see that the elemental moment about the axis *O–O* is $dM = y(\sigma\, dA) = ky^2\, dA$. Thus, the same integral appears when the total moment $M = k \int y^2\, dA$ is evaluated.

A third example is given in Fig. A/1*c*, which shows a circular shaft subjected to a twist or torsional moment. Within the elastic limit of the material, this moment is resisted at each cross section of the shaft by a distribution of tangential or shear stress τ, which is proportional to the radial distance r from the center. Thus, $\tau = kr$, and the total moment about the central axis is $M = \int r(\tau\, dA) = k \int r^2\, dA$. Here the integral differs from that in the preceding two examples in that the area is normal instead of parallel to the moment axis and in that r is a radial coordinate instead of a rectangular one.

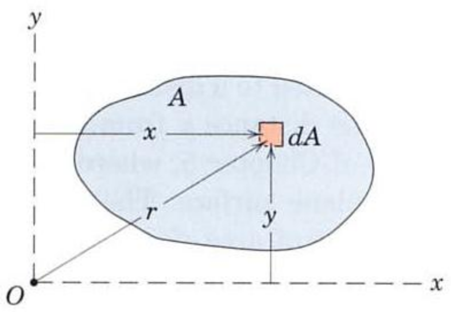
Although the integral illustrated in the preceding examples is generally called the *moment of inertia* of the area about the axis in question, a more fitting term is the *second moment of area*, since the first moment $y\, dA$ is multiplied by the moment arm y to obtain the second moment for the element dA. The word *inertia* appears in the terminology by reason of the similarity between the mathematical form of the integrals for second moments of areas and those for the resultant moments of the so-called inertia forces in the case of rotating bodies. The moment of inertia of an area is a purely mathematical property of the area and in itself has no physical significance.

A/2 Definitions

The following definitions form the basis for the analysis of area moments of inertia.

Rectangular and Polar Moments of Inertia

Consider the area A in the x-y plane, Fig. A/2. The moments of inertia of the element dA about the x- and y-axes are, by definition, $dI_x = y^2\, dA$ and $dI_y = x^2\, dA$, respectively. The moments of inertia of A about the same axes are therefore



$$I_x = \int y^2 \, dA$$

$$I_y = \int x^2 \, dA$$

The moment of inertia of dA about the pole O (z-axis) is, by similar definition, $dI_z = r^2 \, dA$. The moment of inertia of the entire area about O is

$$I_z = \int r^2 \, dA \qquad \textbf{(A/2)}$$

The expressions defined by Eqs. A/1 are called *rectangular* moments of inertia, whereas the expression of Eq. A/2 is called the *polar* moment of inertia.* Because $x^2 + y^2 = r^2$, it is clear that

$$I_z = I_x + I_y \qquad \textbf{(A/3)}$$

Radius of Gyration

Consider an area A, Fig. A/3a, which has rectangular moments of inertia I_x and I_y and a polar moment of inertia I_z about O. We now visualize this area as concentrated into a long narrow strip of area A a distance k_x from the x-axis, Fig. A/3b. By definition the moment of inertia of the strip about the x-axis will be the same as that of the original area if $k_x^2 A = I_x$. The distance k_x is called the *radius of gyration* of the

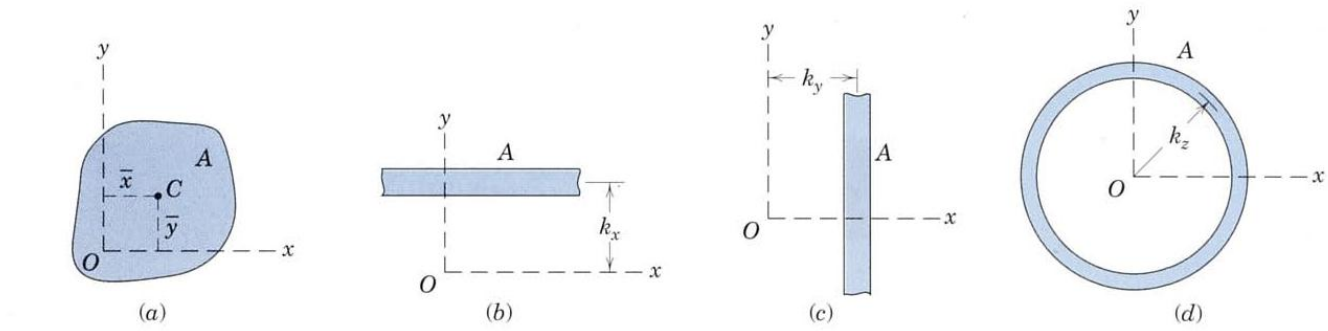


Figure A/3

area about the x-axis. A similar relation for the y-axis is written by considering the area as concentrated into a narrow strip parallel to the y-axis as shown in Fig. A/3c. Also, if we visualize the area as concentrated into a narrow ring of radius k_z as shown in Fig. A/3d, we may express the polar moment of inertia as $k_z{}^2A = I_z$. In summary we write

$$\begin{aligned} I_x &= k_x{}^2A \\ I_y &= k_y{}^2A \\ I_z &= k_z{}^2A \end{aligned} \quad \text{or} \quad \begin{aligned} k_x &= \sqrt{I_x/A} \\ k_y &= \sqrt{I_y/A} \\ k_z &= \sqrt{I_z/A} \end{aligned} \qquad \textbf{(A/4)}$$

The radius of gyration, then, is a measure of the distribution of the area from the axis in question. A rectangular or polar moment of inertia may be expressed by specifying the radius of gyration and the area.

When we substitute Eqs. A/4 into Eq. A/3, we have

$$k_z{}^2 = k_x{}^2 + k_y{}^2 \qquad \textbf{(A/5)}$$

Thus, the square of the radius of gyration about a polar axis equals the sum of the squares of the radii of gyration about the two corresponding rectangular axes.

Transfer of Axes

The moment of inertia of an area about a noncentroidal axis may be easily expressed in terms of the moment of inertia about a parallel centroidal axis. In Fig. A/4 the x_0-y_0 axes pass through the centroid C of the area. Let us now determine the moments of inertia of the area about the parallel x-y axes. By definition, the moment of inertia of the element dA about the x-axis is

$$dI_x = (y_0 + d_x)^2 \, dA$$

Expanding and integrating give us

$$I_x = \int y_0^2 \, dA + 2d_x \int y_0 \, dA + d_x^2 \int dA$$

We see that the first integral is by definition the moment of inertia $\bar{I}_x$ about the centroidal x_0-axis. The second integral is zero, since $\int y_0 \, dA = A\bar{y}_0$ and $\bar{y}_0$ is automatically zero with the centroid on the x_0-axis. The third term is simply Ad_x^2. Thus, the expression for I_x and the similar expression for I_y become

$$I_x = \bar{I}_x + Ad_x^2 \qquad \textbf{(A/6)}$$

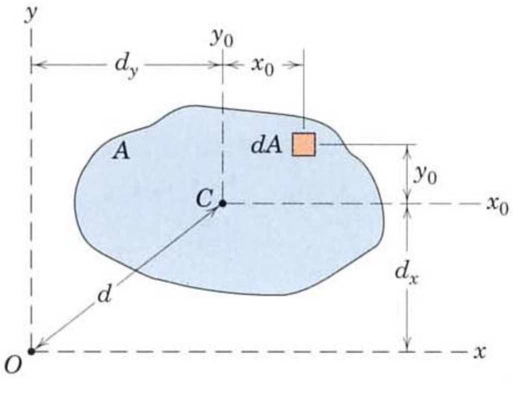
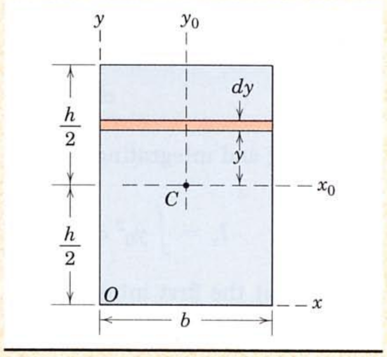


Figure A/4

Sample Problem A/1

Determine the moments of inertia of the rectangular area about the centroidal x_0- and y_0-axes, the centroidal polar axis z_0 through C, the x-axis, and the polar axis z through O.



Solution. For the calculation of the moment of inertia $\bar{I}_x$ about the x_0-axis, a ① horizontal strip of area $b\,dy$ is chosen so that all elements of the strip have the same y-coordinate. Thus,

$$[I_x = \int y^2\,dA] \qquad \bar{I}_x = \int_{-h/2}^{h/2} y^2 b\,dy = \tfrac{1}{12}bh^3 \qquad \textit{Ans.}$$

By interchange of symbols, the moment of inertia about the centroidal y_0-axis is

$$\bar{I}_y = \tfrac{1}{12}hb^3 \qquad \textit{Ans.}$$

The centroidal polar moment of inertia is

$$[\bar{I}_z = \bar{I}_x + \bar{I}_y] \qquad \bar{I}_z = \tfrac{1}{12}(bh^3 + hb^3) = \tfrac{1}{12}A(b^2 + h^2) \qquad \textit{Ans.}$$

By the parallel-axis theorem the moment of inertia about the x-axis is

$$[I_x = \bar{I}_x + Ad_x{}^2] \qquad I_x = \tfrac{1}{12}bh^3 + bh\left(\frac{h}{2}\right)^2 = \tfrac{1}{3}bh^3 = \tfrac{1}{3}Ah^2 \qquad \textit{Ans.}$$

We also obtain the polar moment of inertia about O by the parallel-axis theorem, which gives us

$$[I_z = \bar{I}_z + Ad^2] \qquad I_z = \tfrac{1}{12}A(b^2 + h^2) + A\left[\left(\frac{b}{2}\right)^2 + \left(\frac{h}{2}\right)^2\right]$$

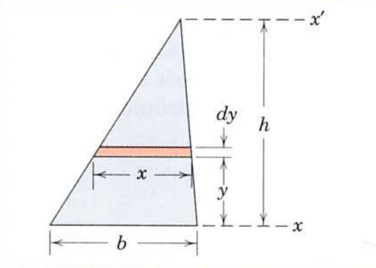
$$I_z = \tfrac{1}{3}A(b^2 + h^2) \qquad \textit{Ans.}$$

Helpful Hint

① If we had started with the second-order element $dA = dx\,dy$, integration with respect to x holding y constant amounts simply to multiplication by b and gives us the expression $y^2b\,dy$, which we chose at the outset.

Sample Problem A/2

Determine the moments of inertia of the triangular area about its base and about parallel axes through its centroid and vertex.



① **Solution.** A strip of area parallel to the base is selected as shown in the figure, ② and it has the area $dA = x\,dy = [(h - y)b/h]\,dy$. By definition

$$[I_x = \int y^2\,dA] \qquad I_x = \int_0^h y^2 \frac{h-y}{h} b\,dy = b\left[\frac{y^3}{3} - \frac{y^4}{4h}\right]_0^h = \frac{bh^3}{12} \qquad \textit{Ans.}$$

By the parallel-axis theorem the moment of inertia $\bar{I}$ about an axis through the centroid, a distance $h/3$ above the x-axis, is

$$[\bar{I} = I - Ad^2] \qquad \bar{I} = \frac{bh^3}{12} - \left(\frac{bh}{2}\right)\left(\frac{h}{3}\right)^2 = \frac{bh^3}{36} \qquad \textit{Ans.}$$

A transfer from the centroidal axis to the x'-axis through the vertex gives

$$[I = \bar{I} + Ad^2] \qquad I_{x'} = \frac{bh^3}{36} + \left(\frac{bh}{2}\right)\left(\frac{2h}{3}\right)^2 = \frac{bh^3}{4} \qquad \textit{Ans.}$$

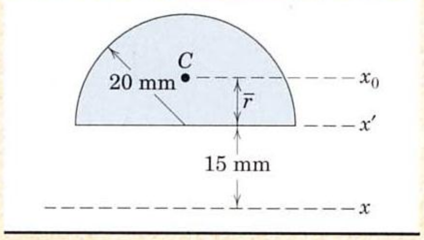
Helpful Hints

① Here again we choose the simplest possible element. If we had chosen $dA = dx\,dy$, we would have to integrate $y^2\,dx\,dy$ with respect to x first. This gives us $y^2x\,dy$, which is the expression we chose at the outset.

② Expressing x in terms of y should cause no difficulty if we observe the proportional relationship between the similar triangles.

Sample Problem A/5

Find the moment of inertia about the x-axis of the semicircular area.



Solution. The moment of inertia of the semicircular area about the x'-axis is one-half of that for a complete circle about the same axis. Thus, from the results of Sample Problem A/3

$$I_{x'} = \frac{1}{2}\frac{\pi r^4}{4} = \frac{20^4\pi}{8} = 2\pi(10^4)\ \text{mm}^4$$

We obtain the moment of inertia $\bar{I}$ about the parallel centroidal axis x_0 next. Transfer is made through the distance $\bar{r} = 4r/(3\pi) = (4)(20)/(3\pi) = 80/(3\pi)$ mm by the parallel-axis theorem. Hence,

$$[\bar{I} = I - Ad^2] \qquad \bar{I} = 2(10^4)\pi - \left(\frac{20^2\pi}{2}\right)\left(\frac{80}{3\pi}\right)^2 = 1.755(10^4)\ \text{mm}^4$$

① Finally, we transfer from the centroidal x_0-axis to the x-axis. Thus,

$$[I = \bar{I} + Ad^2] \qquad I_x = 1.755(10^4) + \left(\frac{20^2\pi}{2}\right)\left(15 + \frac{80}{3\pi}\right)^2$$

$$= 1.755(10^4) + 34.7(10^4) = 36.4(10^4)\ \text{mm}^4 \qquad \textit{Ans.}$$

Helpful Hint

① This problem illustrates the caution we should observe in using a double transfer of axes since neither the x'- nor the x-axis passes through the centroid C of the area. If the circle were complete with the centroid on the x'-axis, only one transfer would be needed.

A/3 Composite Areas

It is frequently necessary to calculate the moment of inertia of an area composed of a number of distinct parts of simple and calculable geometric shape. Because a moment of inertia is the integral or sum of the products of distance squared times element of area, it follows that the moment of inertia of a positive area is always a positive quantity. The moment of inertia of a composite area about a particular axis is therefore simply the sum of the moments of inertia of its component parts about the same axis. It is often convenient to regard a composite area as being composed of positive and negative parts. We may then treat the moment of inertia of a negative area as a negative quantity.

When a composite area is composed of a large number of parts, it is convenient to tabulate the results for each of the parts in terms of its area A, its centroidal moment of inertia $\bar{I}$, the distance d from its centroidal axis to the axis about which the moment of inertia of the entire section is being computed, and the product Ad^2. For any one of the parts the moment of inertia about the desired axis by the transfer-of-axis theorem is $\bar{I} + Ad^2$. Thus, for the entire section the desired moment of inertia becomes $I = \Sigma\bar{I} + \Sigma Ad^2$.

For such an area in the x-y plane, for example, and with the notation of Fig. A/4, where $\bar{I}_x$ is the same as I_{x_0} and $\bar{I}_y$ is the same as I_{y_0} the tabulation would include

Part	Area, A	d_x	d_y	$Ad_x^{\,2}$	$Ad_y^{\,2}$	$\bar{I}_x$	$\bar{I}_y$
Sums	ΣA			$\Sigma Ad_x^{\,2}$	$\Sigma Ad_y^{\,2}$	$\Sigma\bar{I}_x$	$\Sigma\bar{I}_y$

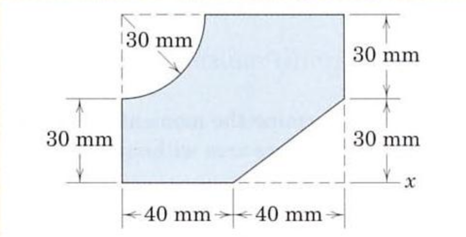
From the sums of the four columns, then, the moments of inertia for the composite area about the x- and y-axes become

$$I_x = \Sigma\bar{I}_x + \Sigma Ad_x^{\,2}$$
$$I_y = \Sigma\bar{I}_y + \Sigma Ad_y^{\,2}$$

Although we may add the moments of inertia of the individual parts of a composite area about a given axis, we may not add their radii of gyration. The radius of gyration for the composite area about the axis in question is given by $k = \sqrt{I/A}$, where I is the total moment of inertia and A is the total area of the composite figure. Similarly, the radius of gyration k about a polar axis through some point equals $\sqrt{I_z/A}$, where $I_z = I_x + I_y$ for x-y axes through that point.

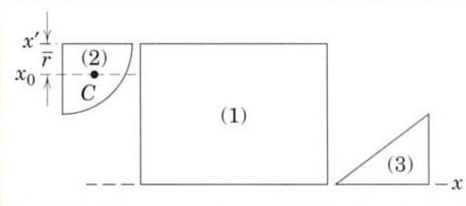
Sample Problem A/7

Calculate the moment of inertia and radius of gyration about the x-axis for the shaded area shown.



Solution. The composite area is composed of the positive area of the rectangle (1) and the negative areas of the quarter circle (2) and triangle (3). For the rectangle the moment of inertia about the x-axis, from Sample Problem A/1 (or Table D/3), is

$$I_x = \frac{1}{3}Ah^2 = \frac{1}{3}(80)(60)(60)^2 = 5.76(10^6)\ \text{mm}^4$$



From Sample Problem A/3 (or Table D/3), the moment of inertia of the negative quarter-circular area about its base axis x' is

$$I_{x'} = -\frac{1}{4}\left(\frac{\pi r^4}{4}\right) = -\frac{\pi}{16}(30)^4 = -0.1590(10^6)\ \text{mm}^4$$

We now transfer this result through the distance $\bar{r} = 4r/(3\pi) = 4(30)/(3\pi) = 12.73$ mm by the transfer-of-axis theorem to get the centroidal moment of inertia of part (2) (or use Table D/3 directly).

① $[\bar{I} = I - Ad^2]$ $\qquad \bar{I}_x = -0.1590(10^6) - \left[-\frac{\pi(30)^2}{4}(12.73)^2\right]$

$$= -0.0445(10^6)\ \text{mm}^4$$

The moment of inertia of the quarter-circular part about the x-axis is now

② $[I = \bar{I} + Ad^2]$ $\qquad I_x = -0.0445(10^6) + \left[-\frac{\pi(30)^2}{4}\right](60 - 12.73)^2$

$$= -1.624(10^6)\ \text{mm}^4$$

Finally, the moment of inertia of the negative triangular area (3) about its base, from Sample Problem A/2 (or Table D/3), is

$$I_x = -\frac{1}{12}bh^3 = -\frac{1}{12}(40)(30)^3 = -0.09(10^6)\ \text{mm}^4$$

The total moment of inertia about the x-axis of the composite area is, consequently,

③ $$I_x = 5.76(10^6) - 1.624(10^6) - 0.09(10^6) = 4.05(10^6)\ \text{mm}^4 \qquad \textit{Ans.}$$

The net area of the figure is $A = 60(80) - \frac{1}{4}\pi(30)^2 - \frac{1}{2}(40)(30) = 3490\ \text{mm}^2$ so that the radius of gyration about the x-axis is

$$k_x = \sqrt{I_x/A} = \sqrt{4.05(10^6)/3490} = 34.0\ \text{mm} \qquad \textit{Ans.}$$

Helpful Hints

① Note that we must transfer the moment of inertia for the quarter-circular area to its centroidal axis x_0 before we can transfer it to the x-axis, as was done in Sample Problem A/5.

② We watch our signs carefully here. Since the area is negative, both $\bar{I}$ and A carry negative signs.

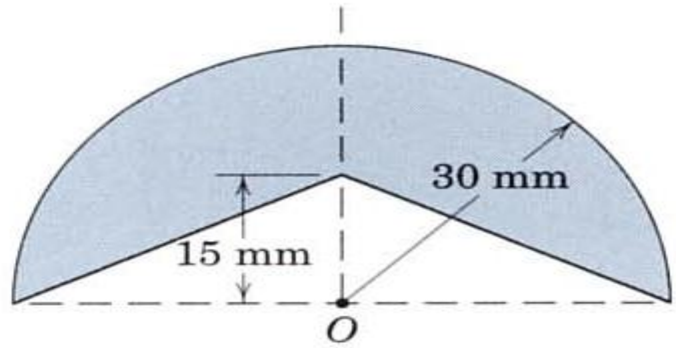
③ If there had been more than the three parts to the composite area, we would have arranged a tabulation of the $\bar{I}$ terms and the Ad^2 terms so as to keep a systematic account of the terms and obtain $I = \Sigma\bar{I} + \Sigma Ad^2$.

Proplems:

1-

A/51 Calculate the polar moment of inertia of the shaded area about point O.

Ans. $I_z = 0.552(10^6)\ \text{mm}^4$



2-

A/43 Determine the moments of inertia of the Z-section about its centroidal x_0- and y_0-axes.

Ans. $\bar{I}_x = 22.6(10^6)\ \text{mm}^4$

$\bar{I}_y = 9.81(10^6)\ \text{mm}^4$